01. a. Hyperbolic cylinder. The generatrices are parallel to z axis.
b. Cone with vertex $V(0,0,0)$. It is a cone with real points (since there are real points other than V, by instance $P(1,1,0)$). It is a surface of revolution around the x axis.
c. Can be written as $x^2 - 3y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$. So it is an hyperboloid of one sheet with center $(0,0,1/2)$. It is a surface of revolution around the line $\{x = 0 \ ; \ z = 1/2\}$.
d. Can be written as $(x-1)^2 + y^2 = (z+1)$. It is an elliptic paraboloid with vertex $V(1,0,-1)$. It is a surface of revolution around the line $\{x = 1 \ ; \ y = 0\}$.
e. Can be written as $(x-1)^2 + y^2 = z^2$. It is a cone with real points and vertex $V(1,0,0)$. It is a surface of revolution around the line $\{x = 1 \ ; \ y = 0\}$.
f. Hyperbolic cylinder. The generatrices are parallel to z axis.
g. Can be written as $z^2 + y^2 = -(x-1)$. It is an elliptic paraboloid (but upside down) with vertex $V(1,0,0)$. It is a surface of revolution around the x axis.
h. Can be written as $(x - 1/4) = -(z + 1/2)^2$. Parabolic cylinder whith generatrices parallel to y axis: All the vertices of the parabolas lie on the line $\{x = 1/4 \ ; \ z + 1/2 = 0\}$.
i. The equation is a function of two linear forms: x and $y+z$. Can be written as $y+z = x^2 - x$. So the lines $\{x = a \ ; \ y + z = a^2 - a\}$ lie on the quadric. Since they all are parallel one to each other, the quadric is a cylinder. The intersection with a simple plane (e.g. $y = 0$ which is not parallel to the generatrices) is a parabola. Therefore the quadric is a parabolic cylinder.
j. Degenerate quadric formed by the union of two planes: $x + y = 0$ and $y = 0$.
k. Degenerate quadric formed by the union of two planes: $x = 0$ e $y + z = 1$.
l. Degenerate quadric formed by the union of two parallel planes: $x - y = \pm i$; since the planes are non-real, the quadric has no real points.

02. a. The equation can be written as $x^2 + 5\left(y^2 + y + \frac{1}{4}\right) + z^2 = 1 + 5 \cdot \frac{1}{4}$

That is $x^2 + 5\left(y + \frac{1}{2}\right)^2 + z^2 = \frac{9}{4}$. So it is an ellipsoid with center $(0, -1/2, 0)$.
b. The quadric is a surface of revolution around the y axis, since its intersection with the planes $y = k$ ($k \in \mathbb{R}$) are circles whose centers lie on a line which is orthogonal to the y axis.
To get the a circle of radius 1, we can intersect the quadric with the planes $y = k$ that are orthogonal to the axis.

$$\begin{cases} x^2 + 5\left(y + \frac{1}{2}\right)^2 + z^2 = \frac{9}{4} \\ y = k \end{cases} \quad \begin{cases} x^2 + z^2 = \frac{9}{4} - 5\left(k + \frac{1}{2}\right)^2 = R^2 \\ y = k \end{cases}$$

Now impose $R^2 = 1$: $\frac{9}{4} - 5\left(k + \frac{1}{2}\right)^2 = 1$. Solving the equation we find $k = -1, 0$.

There are two circles: $\begin{cases} x^2 + z^2 = 1 \\ y = 0 \end{cases} \quad \begin{cases} x^2 + z^2 = 1 \\ y = -1 \end{cases}$

03. a. It is a cone with vertex $(0,0,0)$ since the equation is homogeneous. One can be sure that it is not a degenerate cone since it is easy to find points different from V, by example $P(0,0,1)$.
b. To find the lines it is sufficient to split the equation in this way:
$(2x - 4y)y = -2xz$ Then write it this way:
$\frac{2x - 4y}{x} = \frac{-2z}{y} = k$ and get the lines: $\{2x - 4y = kx \ ; \ -2z = ky\}$. Pay attention to the fact that this way a line is missing: $\{x = 0 \ ; \ y = 0\}$, the z axis.
c. The easiest conics lying on Q are the ones one can get by intersecting Q with planes which are parallel to coordinate planes.
As it is easy to see, the intersection with the plane $x = 1$ is a parabola:
$\begin{cases} 2xy + 2xz - 4y^2 = 0 \\ x = 1 \end{cases} \quad \begin{cases} 2y + 2z - 4y^2 = 0 \\ x = 1 \end{cases}$ This is a parabola, since it is the intersection between a parabolic cylinder and a plane.

d. Let us make some easy calculations

$\begin{cases} 2xy+2xz-4y^2=0 \\ x=1-y \end{cases}$ $\begin{cases} 2(1-y)y+2(1-y)z-4y^2=0 \\ x=1-y \end{cases}$ $\begin{cases} -6y^2-2yz+2y+2z=0 \\ x=1-y \end{cases}$

The cylinder $-6y^2-2yz+2y+2z=0$ can be studied as if it were a conic.

Let us write the two matrices associated to it: $A=\begin{pmatrix} -6 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ Since $\det(A)\neq 0$ the conic is not degenerate. The 2×2 framed matrix B which is associated to the quadratic form has negative determinant.

From these two facts we can infer that the cylinder is hyperbolic and so the intersection of Q with the plane $x+y=1$ is an hyperbola.

04. a. We consider the matrices associated to the quadric: $\det(A)\neq 0$ so the quadric is not degenerate. Now we need to know the eigenvalues of the 3×3 framed matrix B which is associated to the quadratic form.

$$A=\begin{pmatrix} 1 & 1 & 2 & -1 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

To calculate $\det(B-xI)$ it is advantageous to make some elementary operations on the matrix $B-xI$

$$\det\begin{pmatrix} 1-x & 1 & 2 \\ 1 & 1-x & 2 \\ 2 & 2 & -x \end{pmatrix} \overset{[R_2\to R_2-R_1]}{=} \det\begin{pmatrix} 1-x & 1 & 2 \\ x & -x & 0 \\ 2 & 2 & -x \end{pmatrix} \overset{[C_2\to C_2+C_1]}{=} \det\begin{pmatrix} 1-x & 2-x & 2 \\ x & 0 & 0 \\ 2 & 4 & -x \end{pmatrix} =$$

$$=(-x)\det\begin{pmatrix} 2-x & 2 \\ 4 & -x \end{pmatrix}=(-x)(x^2-2x-8)$$

The eigenvalues are 0 , -2 , 4 and so the signature of the quadratic form is $(0,-,+)$. The only quadric with these features is the hyperbolic paraboloid.

b. It is well-known that the hyperbolic paraboloid is a doubly ruled surface. Since the quadric passes through $(0,0,0)$, the tangent plane to Q in this point is the linear form of the equation, that is $-2x=0$.

The intersection can be written this way:

$\begin{cases} x^2+y^2+2xy+4xz+4yz-2x=0 \\ x=0 \end{cases}$ $\begin{cases} y^2+4yz=0 \\ x=0 \end{cases}$ $\begin{cases} y(y+4z)=0 \\ x=0 \end{cases}$

The two lines are therefore: $\begin{cases} y=0 \\ x=0 \end{cases}$ $\begin{cases} y+4z=0 \\ x=0 \end{cases}$

05. a. We consider the matrices associated to the quadric: $\det(A)>0$ so the quadric is not degenerate. Then we calculate the eigenvalues of the 3×3 framed matrix B which is associated to the quadratic form.

$$A=\begin{pmatrix} 3 & 0 & 0 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 1 & 0 \\ 2 & -1 & 0 & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_1=-1$, $\lambda_2=3$, $\lambda_3=3$.

The signature of the quadratic form is $(+,-,-)$ and $\det(A)$ is positive. The only quadric with these features is the hyperboloid of one sheet.

b. The quadric is a revolution surface since there are two coincident eigenvalues of the quadratic form.

To write the axis, first find the center of the quadric by solving the linear system of the partial derivatives of the function $f(x,y,z)$ which defines the quadric.

$$\begin{cases} \partial f/\partial x = 6x+4=0 \\ \partial f/\partial y = 2y+4z-2=0 \\ \partial f/\partial z = 4y+2z=0 \end{cases}$$

The solution is $C(-2/3\ ,\ -1/3\ ,\ 1/3)$

Now we need an eigenvector associated to the "different" eigenvalue, i.e. to $\lambda_1=-1$.

An eigenvector is a solution of the homogeneous system whose matrix of coefficients is $B-\lambda_1 I$. For example $v=(0,1,-1)$.

The axis is the line a through C and parallel to the vector $(0,1,-1)$.

$$a\begin{cases} x=-2/3 \\ y=-1/3+t \\ z=1/3-t \end{cases}$$

c. The first intersection can be written this way:

$\begin{cases} 3x^2+y^2+4yz+z^2+4x-2y=0 \\ y=x \end{cases}$ $\begin{cases} 4x^2+4xz+z^2+2x=0 \\ y=x \end{cases}$ $\begin{cases} (2x+z)^2+2x=0 \\ y=x \end{cases}$

The cylinder $(2x+z)^2+2x=0$ is a parabolic cylinder, since its quadratic form is the square of a linear form, so its intersection with the plane $x = y$ is a parabola .

The tangent plane to Q in this point $(0, 0, 0)$, is the linear form of the equation, that is $4x - 2y = 0$, and, since the quadric has hyperbolic points its intersection with the plane is made out of two lines.

$\begin{cases} 3x^2 + y^2 + 4yz + z^2 + 4x - 2y = 0 \\ 4x - 2y = 0 \end{cases}$ $\begin{cases} 7x^2 + 8xz + z^2 = 0 \\ 4x - 2y = 0 \end{cases}$

Solve the quadratic equation with respect to z: $z_1 = -7x$; $z_2 = -x$.

So $7x^2 + 8xz + z^2 = (z + 7x)(z + x)$. The two lines are: $\begin{cases} z + 7x = 0 \\ 4x - 2y = 0 \end{cases}$ $\begin{cases} z + 7x = 0 \\ 4x - 2y = 0 \end{cases}$

06. a. We consider the matrices associated to the quadric:

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 2 \end{matrix}} & \begin{matrix} 0 \\ 1 \\ -1 \end{matrix} \\ \begin{matrix} 0 & 1 & -1 \end{matrix} & 0 \end{pmatrix}$$

$\det(A) = 0$ so the quadric is degenerate. Then we calculate the eigenvalues of the 3×3 framed matrix B which is associated to the quadratic form.

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1 + \sqrt{2}$, $\lambda_3 = 1 - \sqrt{2}$.

The signature of the quadratic form is $(+, +, -)$. The only quadric with these features is the cone. The vertex is the center of the quadric.

It can be found by solving the linear system of the partial derivatives of the function $f(x, y, z)$ which defines the quadric. The solution is $V(0, 1, 1)$

$$\begin{cases} \partial f/\partial x = 2x = 0 \\ \partial f/\partial y = -2z + 2 = 0 \\ \partial f/\partial z = 4z - 2y - 2 = 0 \end{cases}$$

b. It can be advantageous to translate the coordinates, making $(0, 0, 0)$ the vertex, so the cone will have a homogeneous equation. $\{X = x \; ; \; Y = y - 1 \; ; \; Z = z - 1\}$. By substituting we get $X^2 - 2YZ + 2Z^2 = 0$.

To find the lines it is sufficient to split the equation in this way:

$X \cdot X = Z \cdot (2Y - 2Z)$ Then write it this way: $\dfrac{X}{Z} = \dfrac{2Y - 2Z}{X} = k$ and get the lines: $\{X = kZ \; ; \; 2Y - 2Z = kX\}$ or better $\{x = k(z - 1) \; ; \; 2(y - 1) - 2(z - 1) = kx\}$. Pay attention to the fact that this way a line is missing: $\{z - 1 = 0 \; ; \; x = 0\}$.

c. The three intersection of Q with the coordinate planes are three different conics:

$\begin{cases} x^2 - 2yz + 2z^2 + 2y - 2z = 0 \\ x = 0 \end{cases}$ $\begin{cases} 2yz + 2z^2 + 2y - 2z = 0 \\ x = 0 \end{cases}$ a hyperbola. This can be easily checked by looking at the matrices associated with the conic $2yz + 2z^2 + 2y - 2z = 0$.

$$A = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ 1 & 2 \end{matrix}} & \begin{matrix} 1 \\ -1 \end{matrix} \\ \begin{matrix} 1 & -1 \end{matrix} & 0 \end{pmatrix}$$

Since $\det(A) \neq 0$ the conic is not degenerate. The 2×2 framed matrix B which is associated to the quadratic form has negative determinant. This is enough.

$\begin{cases} x^2 - 2yz + 2z^2 + 2y - 2z = 0 \\ y = 0 \end{cases}$ $\begin{cases} x^2 + 2z^2 - 2z = 0 \\ y = 0 \end{cases}$ clearly an ellipse

$\begin{cases} x^2 - 2yz + 2z^2 + 2y - 2z = 0 \\ z = 0 \end{cases}$ $\begin{cases} x^2 + 2y = 0 \\ z = 0 \end{cases}$ clearly a parabola

07. a. We consider the matrices associated to the quadric:

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{matrix}} & \begin{matrix} 0 \\ 1 \\ k \end{matrix} \\ \begin{matrix} 0 & 0 & k \end{matrix} & 0 \end{pmatrix}$$

$\det(A) = -2k^2$ so the quadric is degenerate if and only if $k = 0$. Then we calculate the eigenvalues of the 3×3 framed matrix B which is associated to the quadratic form.

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 3$.

The signature of the quadratic form is $(+, +, +)$ and $\det(A)$ is not positive. The only quadrics with these features are ellipsoids. It is an ellipsoid having real points if $k \neq 0$, since $\det(A) < 0$. Otherwise it s a degenerate ellipsoid with only one real point.

b. The quadric is always a revolution surface (degenerate if $k = 0$) since there are two coincident eigenvalues of the quadratic form.

The direction of the axis is given by some eigenvector of the other eigenvalue $\lambda_3 = 3$. One solution of the homogeneous system associated to the matrix $B - \lambda_3 I$ is by example $(0, 1, 1)$. Now find the center by solving the linear system of the partial derivatives of the function $f(x, y, z)$ which defines the quadric.

$$\begin{cases} \partial f/\partial x = 2x = 0 \\ \partial f/\partial y = 4y + 2z = 0 \\ \partial f/\partial z = 4z + 2k = 0 \end{cases}$$

The solution is $C(0\ ,\ k/4\ ,\ -k/2)$. From here the axis a.

$$a \begin{cases} x = 0 \\ y = k/4 + t \\ z = -k/2 + t \end{cases}$$

08. We consider the matrices associated to the quadric: $\det(A) = -(k-1)^2$ so the quadric is degenerate if and only if $k = 1$. Then we calculate the eigenvalues of the 3×3 framed matrix B which is associated to the quadratic form.

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & k & 0 & k-1 \\ -1 & 0 & 2 & 0 \\ 0 & k-1 & 0 & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = k$, $\lambda_2 = \dfrac{3+\sqrt{5}}{2} > 0$, $\lambda_3 = \dfrac{3-\sqrt{5}}{2} > 0$. The signature of the quadratic form depends on k.
If $k = 0$ the quadric is an elliptic paraboloid.
If $k = 1$ the quadric is a degenerate ellipsoid (only a point).
If $k > 0$ $(k \neq 1)$ the quadric is an ellipsoid (with real points, since it passes through $(0,0,0)$).
If $k < 0$ the quadric is a hyperboloid. The tangent plane to Q in $(0,0,0)$, is the linear form of the equation, that is $(2k-2)y = 0$. Let us intersect the quadric with this plane.

$$\begin{cases} x^2 + ky^2 - 2xz + 2z^2 + (2k-2)y = 0 \\ y = 0 \end{cases} \qquad \begin{cases} x^2 - 2xz + 2z^2 = 0 \\ y = 0 \end{cases}$$

The cylinder $x^2 - 2xz + 2z^2 = 0$ is clearly degenerate.

Let us write the matrix associated to it: $A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

Since $\det(A) > 0$ the conic is a degenerate ellipse, so the quadric has elliptic point. When $k < 0$ it is always a hyperboloid of two sheets.

09. a. We consider the matrices associated to the quadric: $\det(A) > 0$ so the quadric is not degenerate. Then we calculate the eigenvalues of the 3×3 framed matrix B which is associated to the quadratic form.

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -1$, $\lambda_3 = -1$.
The signature of the quadratic form is $(+,-,-)$ and $\det(A)$ is positive. The only quadric with these features is the hyperboloid of one sheet.

b. It is well-known that the hyperboloid of one sheet is a doubly ruled surface.

To find the lines it is sufficient to split the equation in this way:
$(x-z)(x+z) = -y(y+4x+2)$ Then write it in the two ways:

$$\frac{x-z}{-y} = \frac{y+4x+2}{x+z} = k \qquad \frac{x-z}{y+4x+2} = \frac{-y}{x+z} = k$$

and get the two families of lines:

$$\begin{cases} x - z = -ky \\ y + 4x + 2 = k(x+z) \end{cases} \qquad \begin{cases} x - z = k(y+4x+2) \\ -y = k(x+z) \end{cases}$$

In the first family the line $\{y = 0\ ;\ x+z = 0\}$ is missing
In the second family the line $\{y+4x+2 = 0\ ;\ x+z = 0\}$ is missing.

c. The quadric is a revolution surface since there are two coincident eigenvalues of the quadratic form.
To write its axes first we must find the center of the quadric. It is sufficient to solve the linear system of the partial derivatives of the function $f(x,y,z)$ which defines the quadric.
The solution is $C(-2/3\ ,\ 1/3\ ,\ 0)$

$$\begin{cases} \partial f/\partial x = 2x + 4y = 0 \\ \partial f/\partial y = 2y + 4x + 2 = 0 \\ \partial f/\partial z = -2z = 0 \end{cases}$$

Now we must find all the eigenvectors of B.
$\lambda_1 = 3$. A solution of the homogeneous system whose matrix of coefficients is $B - \lambda_1 I$ is $(1,1,0)$.
$\lambda_2 = \lambda_3 = -1$. The homogeneous system whose matrix of coefficients is $B - \lambda_2 I$ has ∞^2 solutions, since $\lambda_1 = \lambda_2$.

So the main axis (revolution axis) is the the line through C and parallel to the vector $(1,1,0)$.
Since Q is a revolution surface, any line through C and orthogonal to the main axis is a

symmetry axis for Q. So their direction vector must be (a,b,c) s.t. $\langle(a,b,c),(1,1,0)\rangle = 0$, that is $a+b=0$. These vectors are $(a,-a,c)$ Therefore the axes are

$$\begin{cases} x = -2/3+t \\ y = \quad 1/3+t \\ z = \qquad 0 \end{cases} \text{main axis} \qquad \begin{cases} x = -2/3+at \\ y = \quad 1/3-at \\ z = \qquad ct \end{cases} \text{other axes } (a,c \text{ not both } 0)$$

10. a. We consider the matrices associated to the quadric:

$$A = \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & -1 & 0 & 1 \\ -1 & 0 & k & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$\det(A) = -2k$ so the quadric is degenerate if and only if $k=0$. Then we calculate the eigenvalues of the 3×3 framed matrix B which is associated to the quadratic form.
The eigenvalues are $\lambda_1 = 3$, $\lambda_2 = -2$, $\lambda_3 = k$. The signature of the quadratic form depends on k.
If $k=0$ the quadric is a hyperbolic cylinder.
If $k>0$ the quadric is a hyperboloid of two sheets.
If $k<0$ the quadric is a hyperboloid of one sheet.

b. The quadric is a revolution surface when there are two coincident eigenvalues of the quadratic form, that is when $k=3$ or when $k=-2$.

c. We must find an eigenvector for all the eigenvalues of B.
Now we must find all the eigenvectors of B.
A solution of the homogeneous system whose matrix of coefficients is $B-\lambda_1 I$ is $v_1(2,1,0)$.
A solution of the homogeneous system whose matrix of coefficients is $B-\lambda_2 I$ is $v_2(1,-2,0)$.
solution of the homogeneous system whose matrix of coefficients is $B-\lambda_3 I$ is $v_3(0,0,1)$.
Now find the center of the quadric by solving the linear system of the partial derivatives of the function $f(x,y,z)$ which defines the quadric. The solution is $C(-1/3\ ,\ 1/3\ ,\ 0)$

$$\begin{cases} \partial f/\partial x = \qquad 4x+4y = 0 \\ \partial f/\partial y = -2y+4x+2 = 0 \\ \partial f/\partial z = \qquad\qquad 2z = 0 \end{cases}$$

The axes are the lines through C and having respectively v_1, v_2, v_3 as direction vectors.
Normalize the three vectors and write a matrix having the three vector in the columns to get the change of coordinates. But, caution, we have to change the sense on one of the three vectors otherwise the matrix has negative determinant. We change, by example the second.

$$\begin{pmatrix} x+1/3 \\ y-1/3 \\ z \end{pmatrix} = \begin{pmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

11. a. We consider the matrices associated to the quadric:

$$A = \begin{pmatrix} 0 & 0 & 4 & -1 \\ 0 & 0 & 3 & 0 \\ 4 & 3 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

$\det(A) = 9$ so the quadric is not degenerate. Then we calculate the eigenvalues of the 3×3 framed matrix B which is associated to the quadratic form.
The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 5$, $\lambda_3 = -5$.
The signature of the quadratic form is $(0,+,-)$. The only quadric with these features is the hyperbolic paraboloid.

b. To find the lines it is sufficient to split the equation in this way:
$(8x+6y+2)z = 2\cdot x$ Then write it these two ways:

$$\frac{8x+6y+2}{x} = \frac{2}{z} = k \qquad\qquad \frac{8x+6y+2}{2} = \frac{x}{z} = k$$

From the first, we get the first family of lines: $\{8x+6y+2 = kx\ ;\ 2 = kz\}$. Pay attention to the fact that this way a line is missing: $\{x=0\ ;\ z=0\}$, the y axis.
From the second we get second family is $\{8x+6y+2 = 2k\ ;\ x = kz\}$. In this case no line is missing.

c. The main axis of the quadric has the direction of any eigenvector associated with the eigenvalue 0. Solving the homogenous system associated to $B-\lambda_1 I$ we get by example the vector $(3,-4,0)$. Intersecting Q with planes orthogonal to this vectors we get hyperbolas. One of these hyperbolas is degenerate, the one with center in V. Let's study these hyperbolas:

$$\begin{cases} 8xz+6yz-2x+2z = 0 \\ 3x-4y = k \end{cases} \quad \begin{cases} 8xz+6yz-2x+2z = 0 \\ y = (3x-4)/k \end{cases} \quad \begin{cases} 8xz+6(3x-4)z/k-2x+2z = 0 \\ y = (3x-4)/k \end{cases}$$

$\begin{cases} 50xz + 8x + (8-6k)z = 0 \\ y = (3x-4)/k \end{cases}$ Now it can be easily seen that the hyperbola is degenerate only if $k = 4/3$. In this case the center is $\{x = 0\ ;\ z = 0\}$

So the vertex is $V(0, -1/3, 0)$.

d. The axes are the lines passing through V and parallel to the eigenvectors.
A solution of the homogeneous system whose matrix of coefficients is $B - \lambda_2 I$ is $v_1(4, 3, 5)$.
A solution of the homogeneous system whose matrix of coefficients is $B - \lambda_3 I$ is $v_2(-4, -3, 5)$.
The axes are $\begin{cases} x = 3t \\ y = -1/3 - 4t \\ z = 0 \end{cases}$ $\begin{cases} x = 4t \\ y = -1/3 + 3t \\ z = 5t \end{cases}$ $\begin{cases} x = -4t \\ y = -1/3 - 3t \\ z = 5t \end{cases}$

12. a. We consider the matrices associated to the quadric:
$\det(A) = -9$ so the quadric is not degenerate. It is not easy to calculate the eigenvalues of the 3×3 framed matrix B which is associated to the quadratic form.

$$A = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 5 & 5 & 0 \\ 3 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

But we can know their signs by using Sylvester's law of inertia and reducing B both by rows and columns:

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 1 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ C2 \to C_2 - 2C_1 \\ C3 \to C_3 - 3C_1 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & -8 \end{pmatrix} \begin{matrix} R_3 \to R_3 + R_2 \\ C3 \to C_3 + C_2 \end{matrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -9 \end{pmatrix}$$

By Sylvester's law of inertia, the eigenvalues of the reduced matrix have the same signs as those of the matrix B, so the signature of the quadratic form is $(+, +, -)$. Since $\det(A) > 0$, the quadric is a hyperboloid of two sheets.

b. Just substitute the line into the equation of the quadric:
$(t+1)^2 + 5(t-1)^2 + (-t)^2 + 4(t+1)(t-1) + 6(t+1)(-t) + 10(t-1)(-t) = 1$. Solving the equation for t we get $t_1 = -1$ and $t_2 = 1/5$
From here the two points $P_1(0, -2, 1)$ and $P_2(6/5\ ,\ -4/5\ ,\ -1/5)$

c. For sake of simplicity we choose $P_1(0, -2, 1)$.
We find the normal vector of the tangent plane by calculating in P_1 the partial derivatives of the function $f(x, y, z)$ which defines the quadric.
$\begin{cases} \partial f/\partial x = 2x + 4y + 6z & \text{in } P_1:\ 1 \\ \partial f/\partial y = 10y + 4x + 10z & \text{in } P_1:\ 5 \\ \partial f/\partial z = 2z + 6x + 10y & \text{in } P_1:\ 9 \end{cases}$

The tangent plane is therefore $1(x-0) + 5(y+2) + 9(z-1)$. The intersection should be made out of two lines:

$\begin{cases} x^2 + 5y^2 + z^2 + 4xy + 6xz + 10yz = 1 \\ x + 5y + 9z + 1 = 0 \end{cases}$

$\begin{cases} (-5y - 9z - 1)^2 + 5y^2 + z^2 + 4(-5y - 9z - 1)y + 6(-5y - 9z - 1)z + 10yz = 1 \\ x = -5y - 9z - 1 \end{cases}$

$\begin{cases} 5y^2 + 14z^2 + 17yz + 3y + 6z = 0 \\ x = -5y - 9z - 1 \end{cases}$ The cylinder $5y^2 + 14z^2 + 17yz + 3y + 6z = 0$ can be studied as if it were a conic.

First find its center by solving the linear system of the partial derivatives of the function $f(y, z)$ which defines the conic. $\begin{cases} \partial f/\partial y = 10y + 17z + 3 = 0 \\ \partial f/\partial z = 17y + 28z + 6 = 0 \end{cases}$

The solution is $\{y = -2\ ;\ z = 1\}$.

The decompose the quadratic form $5y^2 + 14z^2 + 17yz$ by solving it with respect to y:
$y_1 = -2z\ ;\ y_2 = -7z/5$. So: $5y^2 + 14z^2 + 17yz = 5(y + 2z)(y + 7z/5) = (y + 2z)(5y + 7z)$.
In conclusion: $5y^2 + 14z^2 + 17yz + 3y + 6z = \Big((y+2) + 2(z-1)\Big)\Big(5(y+2) + 7(z-1)\Big) =$
$= (y + 2z)(5y + 7z + 3)$.

The two lines are therefore: $\begin{cases} y + 2z = 0 \\ x + 5y + 9z + 1 = 0 \end{cases}$ $\begin{cases} 5y + 7z + 3 = 0 \\ x + 5y + 9z + 1 = 0 \end{cases}$

21. We need a translation: $\{X = x\ ;\ Y = y-2\ ;\ Z = z-2\}$. The quadric is $(z-2) = \dfrac{x^2}{a^2} + \dfrac{(y-2)^2}{b^2}$.

Since it is a revolution surface, then $a^2 = b^2$ and the equation is $(z-2) = \dfrac{x^2}{a^2} + \dfrac{(y-2)^2}{a^2}$.

To find a^2 we intersect the quadric with the plane $z = 4$.

$\begin{cases} a^2(z-2) = x^2 + (y-2)^2 \\ z = 4 \end{cases}$ $\quad$ $\begin{cases} 2a^2 = x^2 + (y-2)^2 \\ z = 4 \end{cases}$

The circle has radius 1 when $2a^2 = 1$, so we must have $a^2 = 1/2$.
The paraboloid is $z - 2 = 2x^2 + 2(y-2)^2$.

22. The center of the quadric is $C(0,0,2)$, the middle point between V_1 and V_2.
We need only a translation: $\{X = x \; ; \; Y = y \; ; \; Z = z - 2\}$. Since the quadric is a revolution surface, then $a^2 = b^2$ and the equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} - \dfrac{(z-2)^2}{c^2} = -1$.
Intersecting the quadric with the plane $z = 0$ we should get a degenerate ellipse:
$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} - \dfrac{(z-2)^2}{c^2} = -1 \\ z = 0 \end{cases}$ $\quad$ $\begin{cases} \dfrac{x^2}{a^2} + \dfrac{y^2}{a^2} = \dfrac{4}{c^2} - 1 \\ z = 0 \end{cases}$ $\quad$ So we must have $c^2 = 4$.
Now let us substitute P in the equation:
$\dfrac{1}{a^2} + \dfrac{1}{a^2} - \dfrac{9}{4} = -1$. From here easily: $a^2 = \dfrac{8}{5}$. The hyperboloid is $\dfrac{5}{8}x^2 + \dfrac{5}{8}y^2 - \dfrac{(z-2)^2}{4} = -1$

23. The center of the quadric is $C(0,2,0)$ and the main axis is the y axis, so the equation is $\dfrac{x^2}{a^2} + \dfrac{z^2}{a^2} - \dfrac{(y-2)^2}{b^2} = 1$ and a is the radius of the throat circle, that is $a^2 = 4$.
Intersecting the quadric with the plane $y = 0$ we should get a circle of radius 3:
$\begin{cases} \dfrac{x^2}{4} + \dfrac{z^2}{4} - \dfrac{(y-2)^2}{b^2} = 1 \\ y = 0 \end{cases}$ $\quad$ $\begin{cases} x^2 + z^2 = 4\left(\dfrac{4}{b^2} + 1\right) \\ y = 0 \end{cases}$ $\quad$ So we must have $4\left(\dfrac{4}{b^2} + 1\right) = 9$.
We find $b^2 = \dfrac{16}{5}$. The hyperboloid is $\dfrac{x^2}{4} + \dfrac{z^2}{4} - \dfrac{5}{16}(y-2)^2 = 1$

24. We set a new coordinate system where the axis of the parabola is the new Z axis as in the picture. The direction vector of Z axis is $(0,2,1)$, the direction vector of Y axis is $(0,1,-2)$. The X axis is unchanged. So the change is

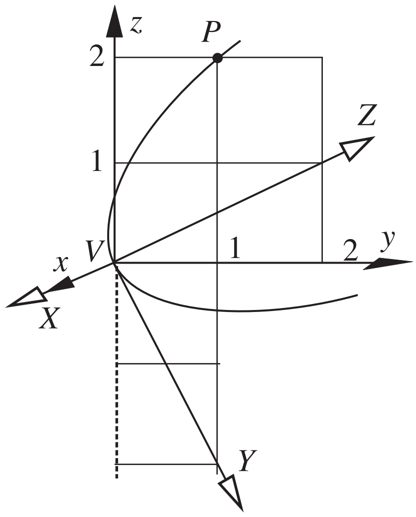
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & 2/\sqrt{5} \\ 0 & -2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \qquad \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{5} & -2/\sqrt{5} \\ 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

In this system the parabola is $\{Z = aY^2 \; ; \; X = 0\}$. Using the inverse change we can find the coordinates of the point P in the new system. They are $(0 \, , \, -3/\sqrt{5} \, , \, 4/\sqrt{5})$.
We substitute the point P and we find $a = 4\sqrt{5}/9$.
So the parabola in new system is $\begin{cases} Z = 4\sqrt{5}\,Y^2/9 \\ X = 0 \end{cases}$

and in the old one: $\begin{cases} \dfrac{2y+z}{\sqrt{5}} = \dfrac{4\sqrt{5}}{9}(y-2z)^2 \\ x = 0 \end{cases}$

The paraboloid is $Z = \dfrac{4\sqrt{5}}{9}Y^2 + \dfrac{4\sqrt{5}}{9}X^2$.

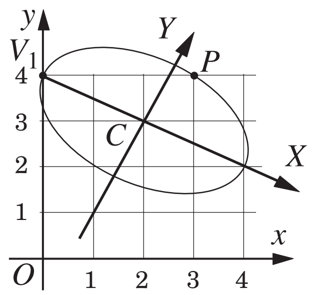
In $Oxyz$ coordinates: $\dfrac{2y+z}{\sqrt{5}} = \dfrac{4\sqrt{5}}{9}(y-2z)^2 + \dfrac{4\sqrt{5}}{9}x^2$



25. a. To find the ellipse in the plane xy it is advisable to set a new coordinate system as in the picture.
The direction vector of X axis is $(2,-1)$, the direction vector of Y axis is $(1,2)$. So the change is

$$\begin{cases} x-2 = (2/\sqrt{5})X + (1/\sqrt{5})Y \\ y-3 = -(1/\sqrt{5})X + (2/\sqrt{5})Y \end{cases}$$

$$\begin{cases} X = (2/\sqrt{5})(x-2) - (1/\sqrt{5})(y-3) = \dfrac{2x-y-1}{\sqrt{5}} \\ Y = (1/\sqrt{5})(x-2) + (2/\sqrt{5})(y-3) = \dfrac{x+2y-8}{\sqrt{5}} \end{cases}$$



In this system the ellipse is $\dfrac{X^2}{5} + \dfrac{Y^2}{b^2} = 1$. Using the inverse change we can find the coordinates of the point P in the new system. They are $(1/\sqrt{5}\,,\, 3/\sqrt{5})$.

We substitute the point P and we find $b^2 = 15/8$. Now we have the ellipse in the original system: $\dfrac{(2x-y-1)^2}{25} + \dfrac{8(x+2y-2)^2}{75} = 1$

In the space the cartesian representation of the ellipse is $\begin{cases} (2x-y-1)^2/25 + 8(x+2y-2)^2/75 = 1 \\ z = 0 \end{cases}$

b. Set $f(x,y) = (2x-y-1)^2/25 + 8(x+2y-2)^2/75$.
The main axis of the cone is parallel to the Z axis so its equation is $f(x,y) - (z-2)^2/c^2 = 0$. Its intersection with the plane $z=0$ should be the ellipse so $f(x,y) - 4/c^2 = 0$ is the ellipse. This means that $-4/c^2 = -1$ and so $c^2 = 4$.
Therefore the cone is $\dfrac{(2x-y-1)^2}{25} + \dfrac{8(x+2y-2)^2}{75} - \dfrac{(z-2)^2}{4} = 0$

26. a. We set a new system of coordinates where:
the X axis is the line from C to V_2
the Y axis is the line from C to V_3.
The direction vectors are
$V_2 - C = (-1,1,0)$ for X axis
$V_3 - C = (-1,-1,3)$ for Y axis
$(V_2-C)\wedge(V_3-C) = (3,3,2)$ for Z axis
In this new system the ellipse is

$$\begin{pmatrix} x-1 \\ y-1 \\ z \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{11} & 3/\sqrt{22} \\ 1/\sqrt{2} & -1/\sqrt{11} & 3/\sqrt{22} \\ 0 & 3/\sqrt{11} & 2/\sqrt{22} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{11} & -1/\sqrt{11} & 3/\sqrt{11} \\ 3/\sqrt{22} & 3/\sqrt{22} & 2/\sqrt{22} \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \\ z \end{pmatrix}$$

$\begin{cases} \dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} = 1 \\ Z = 0 \end{cases}$ where $\begin{array}{l} a = \operatorname{dist}(V_2, C) = \sqrt{2} \\ b = \operatorname{dist}(V_3, C) = \sqrt{11} \end{array}$ and so: $\begin{cases} \dfrac{X^2}{2} + \dfrac{Y^2}{11} = 1 \\ Z = 0 \end{cases}$

The ellipsoid is $\dfrac{X^2}{2} + \dfrac{Y^2}{11} + \dfrac{Z^2}{11} = 1$ So in the $Oxyz$ coordinates we have

$$\begin{cases} X = \dfrac{-x+y}{\sqrt{2}} \\ Y = \dfrac{-x-y+3z+2}{\sqrt{11}} \\ Z = \dfrac{3x+3y+2z-6}{\sqrt{22}} \end{cases}$$

ellipse: $\begin{cases} \dfrac{(-x+y)^2}{4} + \dfrac{(-x-y+3z+2)^2}{121} = 1 \\ 3x+3y+2z-6 = 0 \end{cases}$

ellipsoid:
$$\frac{(-x+y)^2}{4} + \frac{(-x-y+3z+2)^2}{121} + \frac{(3x+3y+2z-6)^2}{242} = 1$$

27. Let us set a new system of coordinates where Z axis is the line from V to C and X axis is any line orthogonal to Z axis.
The direction vector of Z axis is $(C-V) = (-1,-1,1)$. The direction vector of X axis can be any vector orthogonal to $(-1,-1,1)$. For instance $(1,-1,0)$. The direction vector of Y axis is therefore $(-1,-1,1)\wedge(1,-1,0) = (1,1,2)$. So the change of coordinate is

$$\begin{pmatrix} x-1 \\ y-1 \\ z \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \Bigg| \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \\ z \end{pmatrix}$$

In the new coordinates the quadric has equation $Z = \dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2}$. Since it is a revolution surface, then $a^2 = b^2$. In these coordinates the plane containing γ is $Z = d$, where d is the distance between C and V, that is $d = \sqrt{3}$. The intersection is

$\begin{cases} Z = X^2/a^2 + Y^2/a^2 \\ Z = \sqrt{3} \end{cases}$ $\begin{cases} \sqrt{3}a^2 = X^2 + Y^2 \\ Z = \sqrt{3} \end{cases}$ So $\sqrt{3}a^2 = 1$ and $a^2 = 1/\sqrt{3}$.

The paraboloid is:

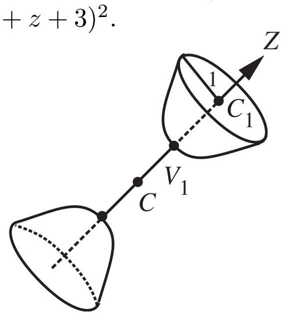
$$\frac{-(x-1)-(y-1)+z}{\sqrt{3}} = \sqrt{3}\frac{((x-1)-(y-1))^2}{2} + \sqrt{3}\frac{((x-1)+(y-1)+2z)^2}{6}$$

28. a. We write the system this way: $\begin{cases} x^2 + y^2 + (z+3)^2 = 9 \\ x - 2y + z = 0 \end{cases}$. So γ is the intersection between the sphere of center $C(0,0,-3)$ and radius $R = 3$ and the plane $x - 2y + z + 3 = 0$. The line through C and orthogonal to the plane is $\{x = t\ ;\ y = -2t\ ;\ z = -3 + t\}$. We intersect this line with the plane and get $t = 1/2$. Hence the point $C_1(1/2\ ,\ -1\ ,\ -5/2)$.
The distance between C and C_1 is $d = \sqrt{3/2}$.
Since $d < R$, then γ is a circle of radius $r = \sqrt{R^2 - d^2} = \sqrt{15/2}$ and center C_1.

b. It is sufficient to eliminate x from the system:
$\begin{cases} x^2 + y^2 + (z+3)^2 = 9 \\ x = 2y - z \end{cases}$ $\begin{cases} (2y-z)^2 + y^2 + (z+3)^2 = 9 \\ x = 2y - z \end{cases}$
The cylinder is $(2y-z)^2 + y^2 + (z+3)^2 = 9$

c. Let us make a simple translation : $Z = z + 3$. This way the vertex is $(0,0,0)$ and the cone will have a homogeneous equation. The circle is $\begin{cases} x^2 + y^2 + Z^2 = 9 \\ x - 2y + Z - 3 = 0 \end{cases}$
Since the cone must contain γ, its equation can be extracted from the system in a simple way: $3 = x - 2y + Z \quad \Rightarrow \quad x^2 + y^2 + Z^2 = (x - 2y + Z)^2$.
The equation is homogeneous and obviously it is verified by every point of γ.
In the previous coordinates: $x^2 + y^2 + (z+3)^2 = (x - 2y + z + 3)^2$.

29. a. Calculate the following vectors:
$(C - C_1) = (2,2,2) \quad (C - V_1) = (1,1,1)$.
Since the vectors are proportional,
the three point are collinear.



b. The point V_1 is located between C and C_1.
We set a new system of coordinates where:
the Z axis is the line from C to V_1
the X axis is any line orthogonal to Z axis.
The direction vectors are:
$V_1 - C = (-1,-1,-1)$ for Z axis
Any vector orthogonal to $(-1,-1,-1)$, for instance $(1,-1,0)$ for X axis
$(-1,-1,-1) \wedge (1,-1,0) = (-1,-1,2)$ for Y axis

$$\begin{pmatrix} x-3 \\ y-2 \\ z-3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 0 & 2/\sqrt{6} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

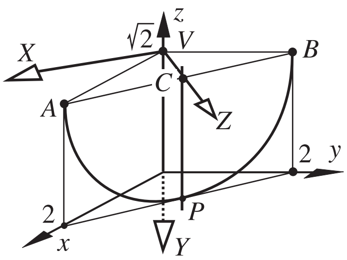
$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ -1/\sqrt{3} & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} x-1 \\ y-1 \\ z \end{pmatrix}$$

In this new system the hyperboloid is $\dfrac{X^2}{a^2} + \dfrac{Y^2}{b^2} - \dfrac{Z^2}{c^2} = -1$. Since it is a revolution surface, then $a^2 = b^2$.
We have $c^2 = 3$, since c is the distance between C and V_1.
The plane of the circle is $Z = d$ where $d = \text{dist}(C, C_1) = 2\sqrt{3}$. So to find a^2 we intersect the quadric with the plane $Z = 2\sqrt{3}$.
$\begin{cases} X^2/a^2 + Y^2/a^2 - Z^2/3 = -1 \\ Z = 2\sqrt{3} \end{cases}$ $\begin{cases} X^2 + Y^2 = 3a^2 \\ Z = 2\sqrt{3} \end{cases}$
The circle has radius 1 when $3a^2 = 1$, so we must have $a^2 = 1/3$.
The hyperboloid is $3\,X^2 + 3\,Y^2 - \dfrac{Z^2}{3} = -1$. By the change of coordinate the quadric is

$$\frac{3((x-3)-(y-2))^2}{2} + \frac{(-(x-3)-(y-2)+2(z-3))^2}{6} - \frac{(-(x-3)-(y-2)-(z-3))^2}{9} = -1$$

30. a. The center of the circle is clearly the point $C(1,1,\sqrt{2})$ (the middle point between A and B). The distance between C and the three points A, B, P is the same: $\sqrt{2}$.
The plane of the circle is clearly $x+y-2=0$ (vertical plane through A and B). So a representation for γ is $\begin{cases}(x-1)^2+(y-1)^2+(z-\sqrt{2})^2=2\\ x+y-2=0\end{cases}$

b. It is sufficient to eliminate x from the system above:
$\begin{cases}(x-1)^2+(y-1)^2+(z-\sqrt{2})^2=2\\ x=2-y\end{cases}$ $\begin{cases}(2-y-1)^2+(y-1)^2+(z-\sqrt{2})^2=2\\ x=2-y\end{cases}$
The cylinder is $2(y-1)^2+(z-\sqrt{2})^2=2$



c. There are two ways of solving the problem:

c_1. Let us set a new system of coordinates where Z axis is the line from V to C, X axis is the line orthogonal to Z axis in the plane of A, B and C and Y axis is the reversed z axis.
The direction vector of Z axis is $(1,1,0)$. The direction vector of X axis is $(1,-1,0)$. The direction vector of Y axis is $(0,0,1)$. So the change of coordinate is

$$\begin{pmatrix}x\\ y\\ z-\sqrt{2}\end{pmatrix}=\begin{pmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2}\\ -1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & -1 & 0\end{pmatrix}\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}\Bigg|\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}=\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2} & 0\\ 0 & 0 & -1\\ 1/\sqrt{2} & 1/\sqrt{2} & 0\end{pmatrix}\begin{pmatrix}x\\ y\\ z-\sqrt{2}\end{pmatrix}$$

In this system the cone is $\dfrac{X^2}{a^2}+\dfrac{Y^2}{a^2}-\dfrac{Z^2}{c^2}=0$ and the plane of A, B and C is $Z=\sqrt{2}$.
Intersect the cone with the plane: $\begin{cases}X^2/a^2+Y^2/a^2-2/c^2=0\\ Z=\sqrt{2}\end{cases}$ We must get a circle of radius $\sqrt{2}$, so $2a^2/c^2=2$, by instance $a=c=1$.
Now substitute $X=x/\sqrt{2}-y/\sqrt{2}$ etc. in the equation and you get the cone:
$$\left(\frac{x-y}{\sqrt{2}}\right)^2+(z-\sqrt{2})^2-\left(\frac{x+y}{\sqrt{2}}\right)^2=0$$

c_2. Write all the lines of the cone:
$\begin{cases}x= & 0+at\\ y= & 0+bt\\ z= & \sqrt{2}+(c-\sqrt{2})t\end{cases}$ From here $\begin{cases}a= & x/t\\ b= & y/t\\ c= & (z-\sqrt{2})/t+\sqrt{2}\end{cases}$
Since (a,b,c) must be a point of the circle, we have $\begin{cases}(a-1)^2+(b-1)^2+(c-\sqrt{2})^2=2\\ a+b-2=0\end{cases}$
Substitute a, b, c in this system: $\begin{cases}(x/t-1)^2+(y/t-1)^2+((z-\sqrt{2})/t)^2=2\\ x/t+y/t-2=0\end{cases}$
Then solve second equation for t and substitute in first: $t=(x+y)/2$
$(2x/(x+y)-1)^2+(2y/(x+y)-1)^2+((2z-2\sqrt{2})/(x+y))^2=2$
The cone is $(x-y)^2+(y-x)^2+(2z-2\sqrt{2})^2=2(x+y)^2$